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An Algorithm for Solving Global Optimization Problems with Nonlinear Constraints

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Abstract. In this paper we propose an algorithm using only the values of the objective function and constraints for solving one-dimensional global optimization problems where both the objective function and constraints are Lipschitzean and nonlinear. The constrained problem is reduced to an unconstrained one by the index scheme. To solve the reduced problem a new method with local tuning on the behavior of the objective function and constraints over different sectors of the search region is proposed. Sufficient conditions of global convergence are established. We also present results of some numerical experiments.

Key words: Global optimization, nonlinear constraints, local tuning, index scheme, global convergence.

1. Introduction

Global optimization problems in different statements are intensively investigated by many authors (see [1] – [15], [20], [25] – [29]). In this paper we consider the global optimization problem with nonlinear constraints

$$min\{g_{m+1}(x) : x \in [a,b], g_j(x) \leq 0, 1 \leq j \leq m\}, \quad (1)$$

where $g_j(x), 1 \leq j \leq m+1$, are multiextremal Lipschitz functions, with the constants $K_j > 0$ i.e.

$$| g_j(x') - g_j(x'') | \leq K_j | x' - x'' |, x', x'' \in [a,b], 1 \leq j \leq m+1. \quad (2)$$

We designate as

$$Q_1 = [a,b], Q_{j+1} = \{x \in Q_j : g_j(x) \leq 0\}, 1 \leq j \leq m, \quad (3)$$

subdomains of the interval $[a,b]$ corresponding to the set of constraints from (1). Thus, we obtain inclusions

$$Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_m \supseteq Q_{m+1}.$$

We shall suppose here in after that the feasible region $Q_{m+1} \neq \leq$ and the subdomains $Q_j, 1 \leq j \leq m+1$, have no isolated points. Using this designations we can rewrite the problem $(1), (2)$ as

$$\min\{g_{m+1}(x) : x \in Q_{m+1}\}. \quad (4)$$

This problem may be reformulated using so-called *index scheme* proposed originally in [10, 23] (see also [22]). The scheme is an alternative to traditional penalty methods. Instead of combining the objective and constraint functions into a penalty one, the scheme considers constraints one at a time at every point where it has been decided to calculate $g_{m+1}(x)$. The constraint i is calculated only if all inequalities

$$g_j(x) \le 0, 1 \le j < i,$$

have been satisfied. In its turn the objective function $g_{m+1}(x)$ is computed only for that points where all constraints have been satisfied. Another important advantage of the index scheme is that it avoids the need to set penalty parameters and the related need to scale the objective function and constraints so that they are commensurate.

Let us present the index scheme. We juxtapose to every point of the interval $[a, b]$ an *index*

$$\nu = \nu(x), 1 \le \nu \le m + 1,$$

which is defined by the conditions

$$g_j(x) \le 0, 1 \le j \le \nu - 1, g_\nu(x) > 0, \tag{5}$$

where for $\nu = m + 1$ the last inequality may be omitted. Consider an auxiliary function $\varphi(x)$ defined over the interval $[a, b]$ as follows

$$\varphi(x) = g_\nu(x) - \begin{cases} 0 & \text{if } \nu(x) < m+1 \\ g^*_{m+1} & \text{if } \nu(x) = m+1 \end{cases} \tag{6}$$

where g^*_{m+1} is the solution of the problem (1), (2) (or of the problem (2)–(4)). Due to (5), (6) the function $\varphi(x)$ has the following properties :

i. $\varphi(x) > 0$, when $\nu(x) < m + 1$;
ii. $\varphi(x) \ge 0$, when $\nu(x) = m + 1$;
iii. $\varphi(x) = 0$, when $\nu(x) = m + 1$ and $g_{m+1}(x) = g^*_{m+1}$.

Thus, the global minimizer of the constrained problem (1), (2) coincides with the solution x^* of the following unconstrained problem

$$\varphi(x^*) = \min\{\varphi(x) : x \in [a, b]\}, \tag{7}$$

and $g_{m+1}(x^*) = g^*_{m+1}$. Obviously g^*_{m+1} is not known. Numerical methods proposed to solve this problem (i.e. to find an estimate of g^*_{m+1}) have been presented in [10], [22], [23].

In this paper for solving the problem (7) we propose a new method based on the information algorithm with local tuning (see [17]) proposed for solving unconstrained problems. The main idea is to tune the algorithm on the objective function and constraints behavior estimating local Lipschitz constants for the functions $g_j(x), 1 \le j \le m + 1$, over different sectors of the search region $[a, b]$. It has been demonstrated in [16]–[18] that using local information can accelerate the global search significantly.

The scheme of the new method is described in the next section. A convergence analysis of the algorithm is contained in Section 3 where sufficient conditions of global convergence are established. Then, we present results of some numerical experiments collected in Section 4. We compare performance of the new method with some algorithms taken from literature. Finally, the last section concludes the paper.

2. Description of the Algorithm

We shall call here in after the operations of choosing a point $x \in [a, b]$, and evaluation $\varphi(x)$ at this point as *iteration of the algorithm.* To start we execute two initial iterations at the points $x^0 = a$ and $x^1 = b$. Suppose now that k iterations have been already done by the method. The choice of the point x^{k+1}, $k \geq 1$, of the $(k+1)$-th iteration is determined by the algorithm presented below.

Step 1. The points $x^0, \dots, x^k$ of the previous k iterations are renumbered by subscripts as follows

$$a = x_0 < x_1 < \dots < x_i < \dots < x_k = b.$$

Step 2. With each point $x_i, 0 \leq i \leq k$, we associate the index $\nu_i = \nu(x_i)$ and the value

$$z_i = g_{\nu_i}(x_i) - \begin{cases} 0 & \text{if } \nu_i < m+1 \\ z^* & \text{if } \nu_i = m+1 \end{cases}$$

where

$$z^* = \min\{g_{m+1}(x_i) : 0 \leq i \leq k, \nu_i = m+1\}.$$

Here the values z_i and z^* estimate the values $\varphi(x_i)$ and g^*_{m+1} from (6).

Step 3. Calculate the low bounds μ_j of the values K_j from (2)

$$\mu_j = \max\{\mid z_p - z_q \mid (x_p - x_q)^{-1} : 0 \leq q < p \leq k, \nu_p = \nu_q = j\},$$

where $1 \leq j \leq m+1$. In all cases when μ_j can not be calculated, set $\mu_j = 0$.

Step 4. For each interval (x_{i-1}, x_i), $1 \leq i \leq k$, calculate the following values

$$M_i = \max\{\lambda_i, \gamma_i\} \tag{8}$$

which estimates the local Lipschitz constant over the interval (x_{i-1}, x_i). The values λ_i and γ_i reflect the influence on M_i of the local and global information obtained in the course of the previous k iterations. The values introduced in (8) are calculated in the following way

$$\lambda_i = \max\{l_i, c_i, r_i\}, \tag{9}$$

where

$$c_i = \begin{cases} \mid z_i - z_{i-1} \mid (x_i - x_{i-1})^{-1} & \text{if } i \geq 1, \nu_i = \nu_{i-1} \\ 0 & \text{otherwise} \end{cases}$$

$$l_i = \begin{cases} \mid z_{i-1} - z_{i-2} \mid (x_{i-1} - x_{i-2})^{-1} & \text{if } i \geq 2, \nu_{i-2} = \nu_{i-1} \geq \nu_i \\ 0 & \text{otherwise} \end{cases}$$

$$r_i = \begin{cases} \mid z_{i+1} - z_i \mid (x_{i+1} - x_i)^{-1} & \text{if } i \leq k-1, \nu_{i+1} = \nu_i \geq \nu_{i-1} \\ 0 & \text{otherwise} \end{cases}$$

The second component of (8) is calculated as

$$\gamma_i = \mu_j(x_i - x_{i-1})/X_j^{\max}, j = \max\{\nu_i, \nu_{i-1}\},$$

where

$$X_j^{\max} = \max\{x_i - x_{i-1} : \max\{\nu_i, \nu_{i-1}\} = j, 1 \leq i \leq k\}.$$

If $M_i < \xi$, set $M_i = \xi$, where $\xi > 0$ is a small number – the parameter of the method reflecting our supposition that the functions $g_j(x), 1 \leq j \leq m+1$, are not constants over the interval (x_{i-1}, x_i).

Step 5. For each interval $(x_{i-1}, x_i), 1 \leq i \leq k$, calculate the *characteristic of the interval*

$$R(i) = \begin{cases} rM_i\Delta_i + (z_i - z_{i-1})^2(rM_i\Delta_i)^{-1} - 2(z_i + z_{i-1}), & \nu_i = \nu_{i-1} \\ 2rM_i\Delta_i - 4z_i, & \nu_i > \nu_{i-1} \\ 2rM_i\Delta_i - 4z_{i-1}, & \nu_{i-1} > \nu_i \end{cases} \quad (10)$$

where

$$\Delta_i = x_i - x_{i-1} \quad (11)$$

and $r > 1$ is a real value – the reliability parameter of the method.

Step 6. Execute the $(k+1)$-th iteration at the point

$$x^{k+1} = (x_t + x_{t-1})/2 - \begin{cases} 0 & \text{if } \nu_t \neq \nu_{t-1} \\ (z_t - z_{t-1})(2rM_t)^{-1} & \text{otherwise} \end{cases} \quad (12)$$

where

$$t = \min\{\arg\max\{R(i) : 1 \leq i \leq k\}\}. \quad (13)$$

Let us comment the algorithm. The information algorithms are derived as optimal statistical decision functions within the framework of a stochastic model representing the function to be optimized as a sample of a random function. The characteristic $R(i)$ in terms of the information approach (see [20, 21]) may be interpreted (after normalizing) as the probability of finding global minimizer at the interval (x_{i-1}, x_i). Speaking in an informal manner we can say that the first item in all three expressions for characteristic in (10) controls the fact that the probability to find a global minimizer is higher in wide intervals. The same is valid for that intervals where the values z_{i-1}, z_i are lower (the last items in the expressions). The second item in the first expression (which is used when both the points x_{i-1}, x_i have have the same index) increases probability of finding a global minimizer for intervals where $(z_i - z_{i-1})^2(rM_i\Delta_i)^{-1}$

The main point of the new algorithm is the following. For every subinterval $(x_{i-1}, x_i), 1 \le i \le k$, we do not use *global* estimates μ_j of the *global* Lipschitz constants $K_j, 1 \le j \le m+1$, from (2) but determine *local* ones $M_i, 1 \le i \le k$, from (8). The values λ_i and γ_i reflect the influence on M_i respectively the local and global information obtained in the course of the previous k iterations. When the interval (x_{i-1}, x_i) is small, then γ_i is small also and due to (8) the local information represented by λ_i has the major importance. We calculate λ_i considering the intervals $(x_{i-2}, x_{i-1}), (x_{i-1}, x_i), (x_i, x_{i+1})$ as that ones which have the strongest influence on the local estimate (see (9)). When the interval (x_{i-1}, x_i) is very wide the local information is not reliable and the global information represented by γ_i is used.

Note that the method proposed here uses the local information over the *whole* search region $[a, b]$ *in the course* of the global search both for the objective function and constraints (being present in a hidden form in the auxiliary function $\varphi(x)$) in contrast with techniques which do it only in a *neighborhood* of local minima *after* stopping their global procedures (see e.g. [15]).

3. Sufficient Conditions of Global Convergence

In this section we demonstrate that the algorithm proposed converges to the global solution of the unconstrained problem (7) and, as consequence, to the global solution of the initial constrained problem (1), (2). To proceed we need the following result.

LEMMA 1. *Let $\bar{x}$ be a limit point of the sequence $\{x^k\}$ generated by the algorithm proposed and $i = i(k)$ be the number of an interval (x_{i-1}, x_i) containing this point in the course of the k-th iteration. Then, for Δ_i from (11) and $R(i)$ from (10) we obtain that*

$$\lim_{k\to\infty} \Delta_{i(k)} = 0 \tag{14}$$

and for every $\delta > 0$ there exists a number $N(\delta)$, such that

$$R(i(k)) < \delta \tag{15}$$

for all $k \ge N(\delta)$.

Proof. The point x^{k+1} from (12) falls into an interval (x_{i-1}, x_i) (where $i(k) = t(k)$ is determined by the formula (13)) and divides this one into two subintervals

$$(x_{i-1}, x^{k+1}), (x^{k+1}, x_i)$$

for which the following inequality

$$\max\{(x_i - x^{k+1}), (x^{k+1} - x_{i-1})\} \leq \alpha(x_i - x_{i-1}) \tag{16}$$

holds when $0.5 \leq \alpha < 1$. In the case $\nu_t \neq \nu_{t-1}$ we obtain (16) immediately from (12) taking $\alpha = 0.5$. In the opposite case from (8) and (9) it follows that

$$| z_t - z_{t-1} | \leq M_t(x_t - x_{t-1}), \nu_t = \nu_{t-1}. \tag{17}$$

From this inequality, (12) and the fact that $r > 1$ we can conclude that (16) is true when

$$\alpha \leq (r+1)/(2r) < 1.$$

Thus, (16) has been proved. Now, considering (16) together with existence of a sequence converging to $\bar{x}$ (this point is a limit point of $\{x^k\}$) we can deduce that (14) holds. Note, that in the case when two intervals containing the point $\bar{x}$ there exist (i.e. when $\bar{x} \in \{x^k\}$) the number $i = i(k)$ is juxtaposed to the interval for which (14) takes place.

Let us demonstrate (15). From (10) and (17) we obtain

$$R(i(k)) \leq 2\Delta_i r M_i,$$

taking into account that $\varphi(x) \geq 0$. This inequality and (14) lead to (15), because $M_i \leq K_j$, where K_j is from (2) and $j = \nu_i$. Lemma has been proved.

THEOREM 1. *Let x^* be any solution of the problem (7) and $j = j(k)$ be the number of an interval (x_{j-1}, x_j) containing this point in the course of the k-th iteration. Then, if for $k \geq k^*$ the condition*

$$rM_j > \begin{cases} C_j + \sqrt{C_j^2 - D_j^2} & \text{if } \nu_{j-1} = \nu_j \\ 2C_j & \text{if } \nu_{j-1} \neq \nu_j \end{cases} \tag{18}$$

takes place, the point x^* will be a limit point of $\{x^k\}$. The values C_j and D_j used in (18) are defined as follows

$$C_j = \begin{cases} z_{j-1}/(x^* - x_{j-1}) & \text{if } \nu_{j-1} > \nu_j \\ \max\{z_{j-1}/(x^* - x_{j-1}), z_j/(x_j - x^*)\} & \text{if } \nu_{j-1} = \nu_j \\ z_j/(x_j - x^*) & \text{if } \nu_{j-1} < \nu_j \end{cases} \tag{19}$$

$$D_j = \begin{cases} | z_j - z_{j-1} | /(x_j - x_{j-1}) & \text{if } \nu_{j-1} = \nu_j \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Proof. Consider the case $\nu_{j-1} = \nu_j$. Due to (19) we can write

$$z_{j-1} \leq C_j(x^* - x_{j-1}), \tag{21}$$

$$z_j \leq C_j(x_j - x^*). \tag{22}$$

Then, from (21) and (22) we obtain

$$z_j + z_{j-1} \le C_j(x_j - x_{j-1}).$$

Let us estimate the characteristic $R(j(k))$ of the interval (x_{j-1}, x_j). Using (20) and the last inequality we deduce

$$R(j(k)) \ge (x_j - x_{j-1})(rM_j + D_j^2(rM_j)^{-1} - 2C_j).$$

Now, due to (18) we can conclude that

$$R(j(k)) > 0. \tag{23}$$

In the case $\nu_{j-1} > \nu_j$ the estimate (21) takes place due to (19). From (10) it follows that

$$R(j(k)) \ge 2(x_j - x_{j-1})(rM_j - 2C_j),$$

and, consequently, taking into consideration (18) the inequality (23) holds in this case also. Truth of (23) for $\nu_{j-1} < \nu_j$ is demonstrated by analogy.

Assume now, that x^* is not a limit point of the sequence $\{x^k\}$. Then, there exists a number Q such that for all $k \ge Q$ the interval $(x_{j-1}, x_j), j = j(k)$, is not changed, i.e. new points will not fall into this interval.

Consider again the interval (x_{i-1}, x_i) from Lemma 1 containing a limit point $\bar{x}$. It follows from (15) that there exists a number N such that

$$R(i(k)) < R(j(k))$$

for all $k \ge k^* = \max\{Q, N\}$. This means that starting from k^* the characteristic of the interval $(x_{i-1}, x_i), i = i(k), k \ge k^*$, is not maximal. Thus, a trial will fall into the interval (x_{j-1}, x_j). But this fact contradicts to our assumption that x^* is not a limit point.

4. Numerical Experiments

For conducting numerical experiments we have constructed 6 test problems with multiextremal objective functions and constraints using functions proposed in [7] for testing one-dimensional global optimization algorithms. All test problems (each problem i is presented in the corresponding figure $i, 1 \le i \le 6$) have a single constraint. In all the problems solution of the constrained problem is unique and does not coincide with the unconstrained one. The objective functions in the figures are indicated by the arrow and the letters "OF". Admissible regions are shown by the thick lines.

As objective functions and constraints have different domains of definition we have transform some of them to inscribe in the region $a \le x \le b, \min \le f(x) \le \max$, where the values $a, b, \min, \max$ are defined for every function. Below we present functions taken from [7] and then their transformations.

Problem 1. Objective function : $f(x) = (3x - 1.4)\sin 18x, x \in [0, 1.2]$.
Transformation : $a = 0, b = 1.2, \min = -2, \max = 1.5$.
Constraint : $g(x) = -e^{-x}\sin 2\pi x, x \in [0, 4]$.
Transformation : $a = 0, b = 1.2, \min = -1.5, \max = 1.2$.
Solution : $x^* = 0.31, z^* = -0.289$.

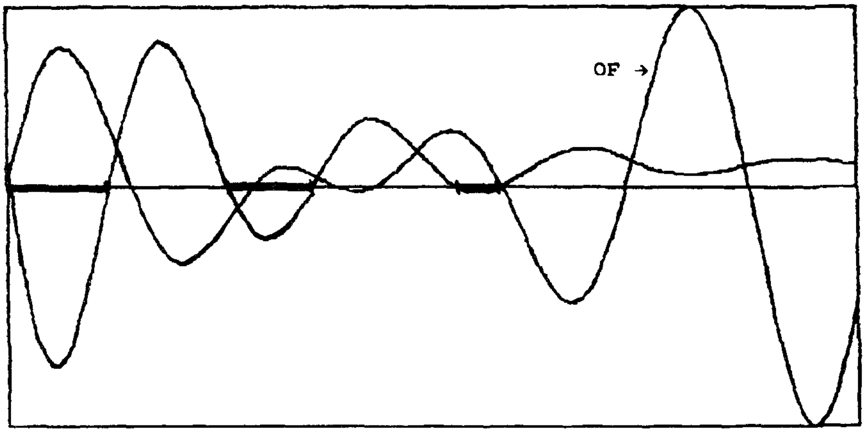

Fig. 1 Problem 1.

Problem 2. Objective function : $f(x) = -\sum_{k=1}^{5} k\sin((k+1)x + k), x \in [-10, 10]$.
Transformation : $a = -10, b = 10, \min = -12.03, \max = 14.84$.
Constraint : $g(x) = \frac{1}{6}x^6 - \frac{52}{25}x^5 + \frac{39}{80}x^4 + \frac{71}{10}x^3 - \frac{79}{20}x^2 + 0.1, x \in [-1.5, 11]$.
Transformation : $a = -10, b = 10, \min = -1, \max = 5$.
Solution : $x^* = 9.81, z^* = -3.735$.

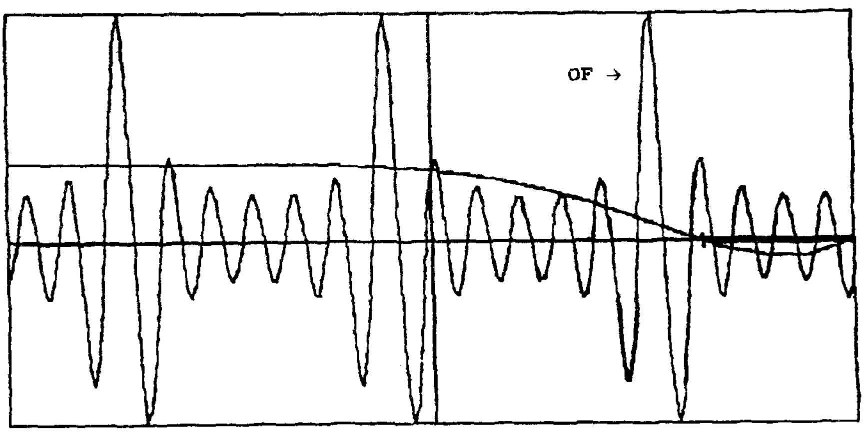

Fig. 2. Problem 2.

Problem 3. Objective function : $f(x) = -\sum_{k=1}^{5} k\cos((k+1)x+k), x \in [-10, 10]$.
Transformation : $a = -10, b = 10, \min = -3, \max = 4$.
Constraint : $g(x) = -e^{-x}\sin 2\pi x, x \in [0, 4]$.
Transformation : $a = -10, b = 10, \min = -1.5, \max = 1.5$.
Solution : $x^* = -8.29, z^* = -0.867$.

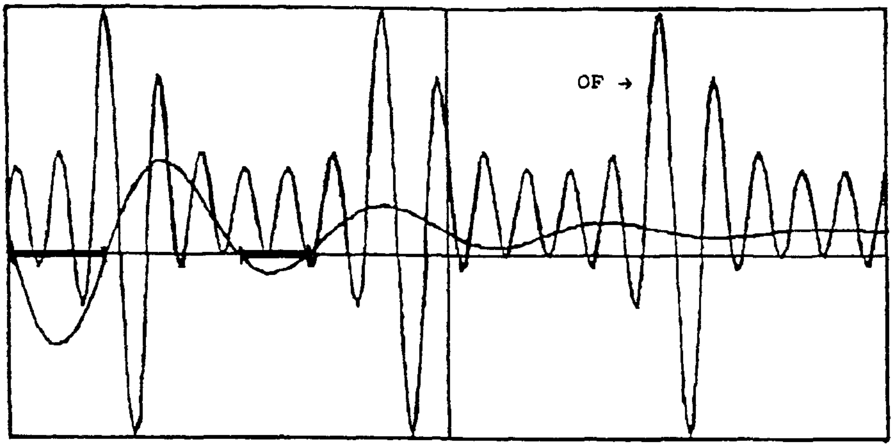

Fig. 3 Problem 3.

Problem 4. Objective function : $f(x) = -x + \sin 3x - 1, x \in [0, 6.5]$.
Transformation : $a = -10, b = 10, \min = -7.816, \max = -0.468$.
Constraint : $g(x) = -e^{-x^2}(x + \sin x), x \in [-10, 10]$.
Transformation : $a = -10, b = 10, \min = -4, \max = 6$.
Solution : $x^* = 1.63, z^* = -5.721$.

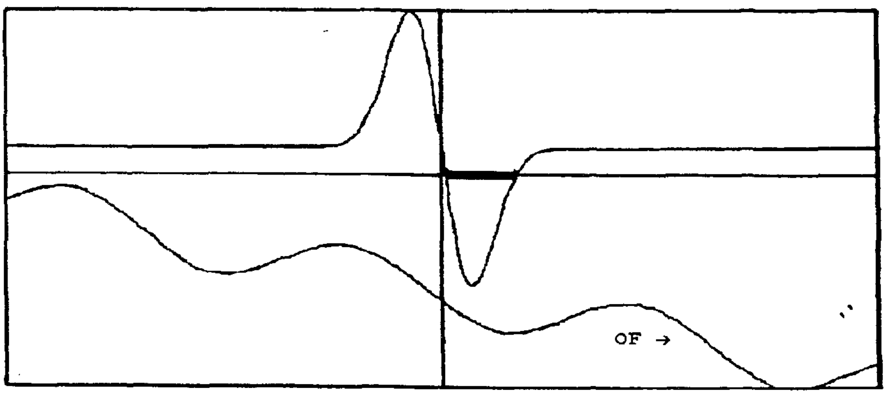

Fig. 4. Problem 4.

Problem 5. Objective function : $f(x) = -(x - \sin x)e^{-x^2}, x \in [-10, 10]$.
Transformation : $a = -10, b = 10, \min = -2, \max = 2$.
Constraint : $g(x) = -e^{-x^2}(x + \sin x), x \in [-10, 10]$.
Transformation : $a = -10, b = 10, \min = -4, \max = 6$.
Solution : $x^* = 1.2, z^* = -2$.

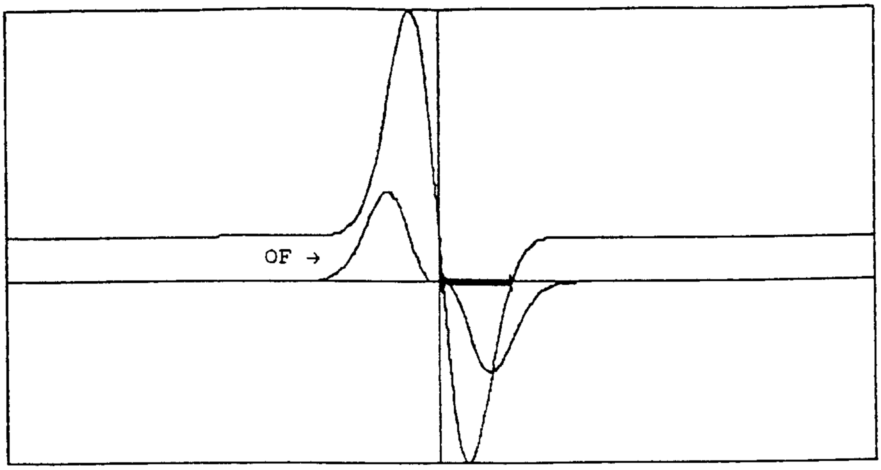

Fig. 5 Problem 5.

Problem 6. Objective function : $f(x) = \sin x + \sin \frac{2}{3}x, x \in [3.1, 20.4]$.
Transformation : $a = 3.1, b = 20.4, \min = -2, \max = 6$.
Constraint : $f(x) = -\sum_{k=1}^{5} k\cos((k+1)x + k), x \in [-10, 10]$.
Transformation : $a = 3.1, b = 20.4, \min = -3, \max = 12.86$.
Solution : $x^* = 16.65, z^* = -1.78$.

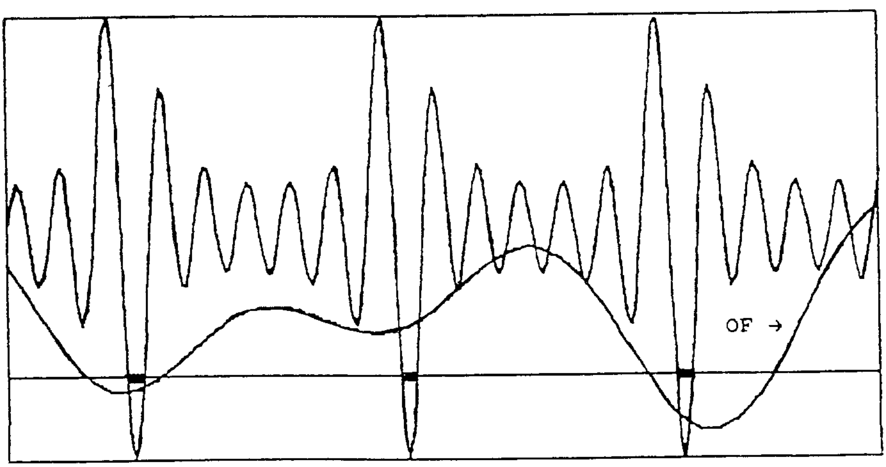

Fig. 6. Problem 6.

TABLE I. Results of numerical experiments.

Problem	PM	IA	New
1	200*	75	65
2	129	89	61
3	162*	140	89
4	200	200	61
5	200	196	148
6	200	200	106
Average	> 181.83	> 136.67	88.33

We compare the new algorithm with two methods taken from literature and belonging to the class of information global optimization procedures. First of them is the basic information algorithm from [22] proposed for solving unconstrained problems. Our constrained problems are reduced to the unconstrained ones by the following penalty function

$$F(x) = f(x) + P \max\{0, g(x)\},$$

where P is a penalty coefficient. This method is indicated by "PM" in Table 1 containing numbers of trials executed by the algorithms before satisfaction of the stopping rule. The second method tested is the index algorithm proposed in [23]. It is indicated in Table 1 as "IA".

We have chosen the following parameters for the algorithms. For PM reliability parameter $r = 2$ and $P = 5$. In the cases PM has not converged to the global solution of the constrained problem for $P = 5$ we used $P = 10$ and included in Table 1 results only for $P = 10$. These cases are indicated in the table by "*". The reliability parameters for the index algorithm and the new method have been also chosen $r = 2$. The parameter $\xi = 10^{-6}$.

We stopped the search when the length of the interval where the next trial point should be executed was less than $\epsilon = 0.001$. If number of trials executed by a method exceeded 200 we stopped the search also.

Results of numerical comparison presented in Table 1 demonstrate that the new algorithm solves problems 1–6 faster than the other methods tested.

5. Conclusions

A new algorithm for solving one-dimensional Lipschitz global optimization problems with nonlinear constraints has been proposed in the paper. The index scheme has been applied to reduce the constrained problem to an unconstrained one. The main idea of the new method was to use local information about the behavior of the objective function and constraints over different sectors of the search region. Sufficient conditions of global convergence have been established for the algorithm.

Numerical experiments executed demonstrate quite satisfactory performance in comparison with the other techniques tested.

The one-dimensional algorithm proposed here can be generalized to the multi-dimensional case following [17] and to the case of parallel computations (see [19, 24]).

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